Government College of Engineering and Research
Avasari, Pune

Fundamental of Finite Element Analysis

Mr. Sanjay D. Patil

Assistant Professor,

Automobile Department

sanjaypatil365@gmail.com

Unit 4

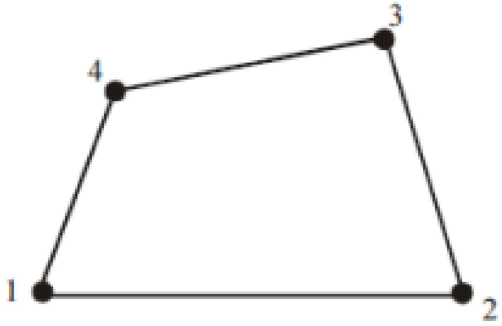
Isoparametric Elements & Numerical Integration

Iso-parametric Element

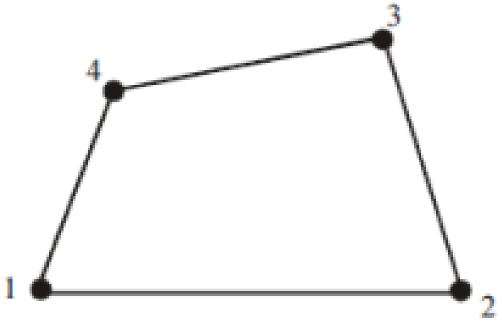
- The element in which the field variables and physical variable are approximated in the same way are called as iso-parametric element.

$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4$$
$$v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4$$

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$
$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$$



For field variable



For physical variable

Sub-parametric Element

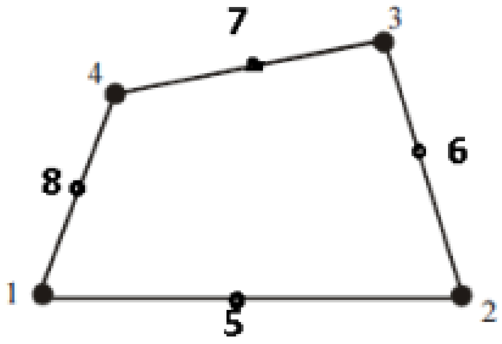
- The element in which the field variables are approximated by higher degree polynomial than physical are called as iso-parametric element.

$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4 + N_5u_5 + N_6u_6 + N_7u_7 + N_8u_8$$

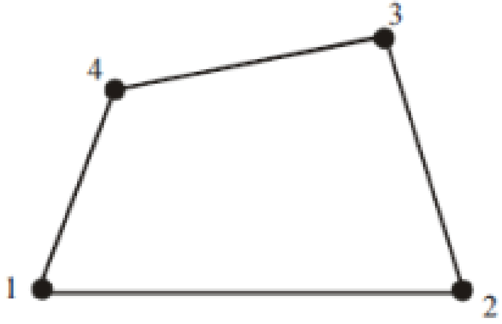
$$v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4 + N_5v_5 + N_6v_6 + N_7v_7 + N_8v_8$$

$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4$$

$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4$$



For field variable

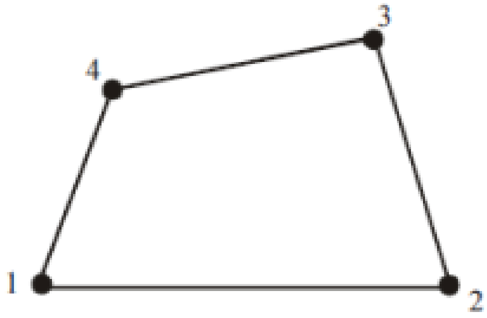


For physical Variable

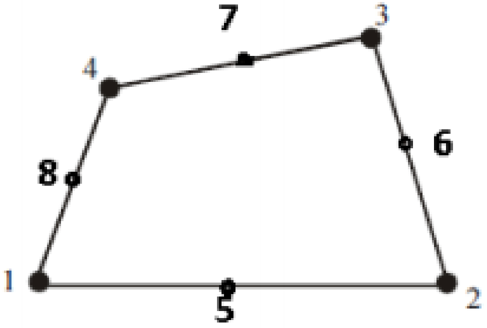
Super-parametric Element

- The element in which the field variables are approximated by lower degree polynomial than physical are called as super-parametric element.

$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4$$
$$v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4$$
$$x = N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4 + N_5x_5 + N_6x_6 + N_7x_7 + N_8x_8$$
$$y = N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4 + N_5y_5 + N_6y_6 + N_7y_7 + N_8y_8$$



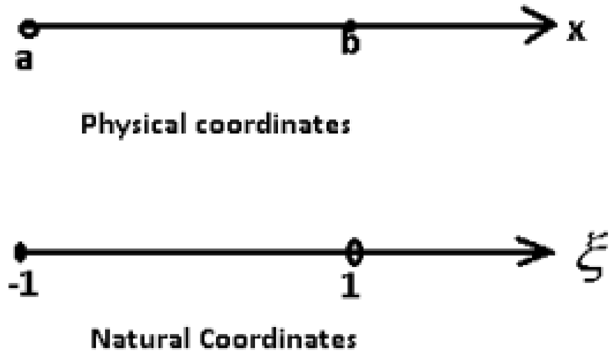
For Field Variable



For Physical Variable

Jacobian Matrix:

- It is used for transforming natural coordinates to global coordinates and vice versa.
- It is defined as the ratio of the sizes of the element in physical and natural coordinate system.
- In one dimension, it is a ratio of infinitesimal length in the physical coordinates to the corresponding mapped length in natural coordinates.



$$x = \frac{a+b}{2} + \frac{b-a}{2}\xi$$

$$dx = \frac{b-a}{2} d\xi$$

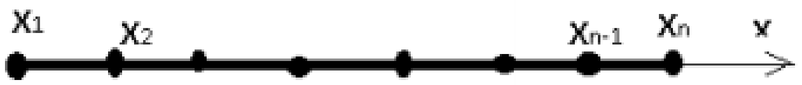
$$dx = J d\xi \qquad J = \frac{b-a}{2}$$

- In two dimension, the determinant of the jacobian matrix is the ratio of infinitesimal area in physical coordinate to that in natural coordinates.

- In three dimension, the determinant of the jacobian matrix is the ratio of infinitesimal volume in physical coordinate to that in natural coordinate system.

2 Noded Bar Element

- The shape function can be derived using lagrangean polynomial for one dimensional and can be extended to 2 and 3 dimensional problem.



$$N_n(x) = \frac{x - x_1 \quad x - x_2 \quad .... \quad x - x_{n-1}}{x_n - x_1 \quad x_n - x_2 \quad .... \quad x_n - x_{n-1}}$$

For 2 noded bar element:

$$N_n(x) = \frac{x - x_1 \quad x - x_2 \quad \ldots\ldots \quad x - x_{n-1}}{x_n - x_1 \quad x_n - x_2 \quad \ldots\ldots \quad x_n - x_{n-1}}$$

$$N_1(\xi) = \frac{x - x_2}{x_1 - x_2} = \frac{\xi - 1}{-1 - 1} = \frac{1 - \xi}{2}$$

$$N_2(\xi) = \frac{x - x_1}{x_2 - x_1} = \frac{\xi - (-1)}{1 - (-1)} = \frac{1 + \xi}{2}$$

For 3 noded bar element:

$$N_n(x) = \frac{x - x_1 \quad x - x_2 \quad .... \quad x - x_{n-1}}{x_n - x_1 \quad x_n - x_2 \quad .... \quad x_n - x_{n-1}}$$

$$N_1(\xi) = \frac{x - x_2 \quad x - x_3}{x_1 - x_2 \quad x_1 - x_3} = \frac{\xi - 0 \quad \xi - 1}{-1 - 0 \quad -1 - 1} = \frac{\xi \;\; 1 - \xi}{2}$$

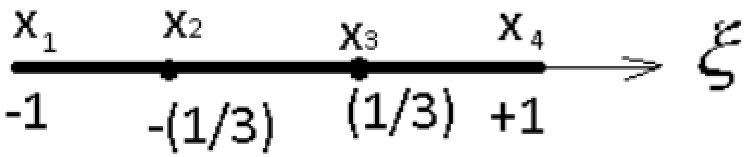
$$N_2(\xi) = \frac{x - x_1 \quad x - x_3}{x_2 - x_1 \quad x_2 - x_3} = \frac{\xi - (-1) \quad \xi - 1}{0 - (-1) \quad 0 - 1} =$$

$$\frac{1 + \xi \quad \xi - 1}{-1} = 1 - \xi^2$$

For 3 noded bar element:

$$N_3(\xi) = \frac{x - x_1}{x_3 - x_1}\,\frac{x - x_2}{x_3 - x_2} = \frac{\xi - (-1)}{1 - (-1)}\,\frac{\xi - 0}{1 - 0} = \frac{1 + \xi}{2}\,\xi$$

For 4-noded element:



$$N_n(x) = \frac{x - x_1 \quad x - x_2 \quad \ldots. \quad x - x_{n-1}}{x_n - x_1 \quad x_n - x_2 \quad \ldots. \quad x_n - x_{n-1}}$$

$$N_1(\xi) = \frac{x - x_2 \quad x - x_3 \quad x - x_4}{x_1 - x_2 \quad x_1 - x_3 \quad x_1 - x_4} =$$

$$\frac{\xi - (-1/3) \quad \xi - 1/3 \quad \xi - 1}{-1 - (-1/3) \quad -1 - 1/3 \quad -1 - 1} =$$

$$\frac{9}{16} \; 1 - \xi \left(\xi^2 - \frac{1}{9} \right)$$

$$N_2(\xi) = \frac{x - x_1 \quad x - x_3 \quad x - x_4}{x_2 - x_1 \quad x_2 - x_3 \quad x_2 - x_4} =$$

$$\frac{\xi - (-1) \quad \xi - 1/3 \quad \xi - 1}{(-1/3) - (-1) \quad (-1/3) - 1/3 \quad (-1/3) - 1} =$$

$$\frac{27}{16} \; \xi^2 - 1 \left(\xi - \frac{1}{3} \right)$$

$$N_3(\xi) = \frac{x - x_1 \quad x - x_2 \quad x - x_4}{x_3 - x_1 \quad x_3 - x_2 \quad x_3 - x_4} =$$

$$\frac{\xi - (-1) \quad \xi - (-1/3) \quad \xi - 1}{(1/3) - (-1) \quad (1/3) - (-1/3) \quad (1/3) - 1} =$$

$$\frac{27}{16} \; 1 - \xi^2 \; \left(\xi + \frac{1}{3}\right)$$

$$N_4(\xi) = \frac{x - x_1 \quad x - x_2 \quad x - x_3}{x_4 - x_1 \quad x_4 - x_2 \quad x_4 - x_3} =$$

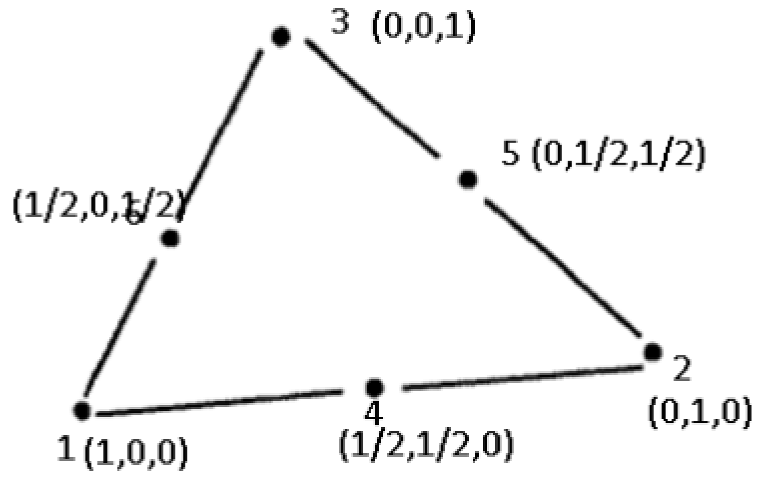
$$\frac{\xi - (-1) \quad \xi - (-1/3) \quad \xi - 1/3}{1 - (-1) \quad 1 - (-1/3) \quad 1 - 1/3} =$$

$$\frac{9}{16} \; 1 + \xi \; \left(\xi^2 - \frac{1}{9} \right)$$

6-Noded Triangular Element

In natural coordinate the shape function for ith node can be written as

$$N_i = a_i\xi_1^2 + b_i\xi_2^2 + c_i\xi_3^2 + d_i\xi_1\xi_2 + e_i\xi_2\xi_3 + f_i\xi_1\xi_3 \quad \text{-------1}$$



6-noded triangular element

For deriving the shape function for N1

$$N_1 = a_1\xi_1^2 + b_1\xi_2^2 + c_1\xi_3^2 + d_1\xi_1\xi_2 + e_1\xi_2\xi_3 + f_1\xi_1\xi_3 \quad \text{-------2}$$

The shape function has 0 value at all other node except 1, where it is 1.

At node 2, $\xi_1 = \xi_3 = 0, \xi_2 = 1$

$$b_1 = 0$$

At node 3, $\xi_1 = \xi_2 = 0, \xi_3 = 1$

$$c_1 = 0$$

At node 5, $\xi_2 = \xi_3 = (1/2), \xi_1 = 0$

$$e_1 = 0$$

At node 1, $\xi_2 = \xi_3 = 0, \xi_1 = 1$

$$a_1 = 1$$

At node 4, $\xi_1 = \xi_2 = (1/2), \xi_3 = 0$

$$d_1 = -1$$

At node 6, $\xi_1 = \xi_3 = (1/2), \xi_2 = 0$

$$f_1 = -1$$

Substituting the values in equation (2) gives,

$$N_1 = \xi_1^2 - \xi_1\xi_2 - \xi_1\xi_3$$

$$N_1 = \xi_1^2 - \xi_1(\xi_2 + \xi_3)$$

$$N_1 = \xi_1(2\xi_1 - 1) \qquad \text{-----3}$$

Similarly the shape function for the N2 and N3 can be written as

For N2

$$N_2 = a_2\xi_1^2 + b_2\xi_2^2 + c_2\xi_3^2 + d_2\xi_1\xi_2 + e_2\xi_2\xi_3 + f_2\xi_1\xi_3 \quad \text{-----4}$$

Finding the unknown a2, b2, c2, d2, e2 and f2 and substituting in equation (4) gives

$$N_2 = \xi_2(2\xi_2 - 1) \quad \text{-------5}$$

For N3 $N_3 = a_3\xi_1^2 + b_3\xi_2^2 + c_3\xi_3^2 + d_3\xi_1\xi_2 + e_3\xi_2\xi_3 + f_3\xi_1\xi_3$ -----6

Finding the unknown a3, b3, c3, d3, e3 and f3 and substituting in equation (6) gives

$$N_3 = \xi_3(2\xi_3 - 1) \quad \text{-------7}$$

For deriving the expression for N4

$$N_4 = a_4\xi_1^2 + b_4\xi_2^2 + c_4\xi_3^2 + d_4\xi_1\xi_2 + e_4\xi_2\xi_3 + f_4\xi_1\xi_3 \text{ -------8}$$

The shape function has 0 value at all other node except 4, where it is 1.

At node 1, $\xi_2 = \xi_3 = 0, \xi_1 = 1$

$$a_4 = 0$$

At node 2, $\xi_1 = \xi_3 = 0, \xi_2 = 1$

$$b_4 = 0$$

At node 3, $\xi_1 = \xi_2 = 0, \xi_3 = 1$

$$c_4 = 0$$

At node 5, $\xi_2 = \xi_3 = (1/2), \xi_1 = 0$

$$e_4 = 0$$

At node 6, $\xi_1 = \xi_3 = (1/2), \xi_2 = 0$

$$f_4 = 0$$

At node 4, $\xi_1 = \xi_2 = (1/2), \xi_3 = 0$

$$d_4 = 4$$

Substituting the values in equation (8) gives,

$$N_4 = 4\xi_1\xi_2 \qquad \text{-----9}$$

Similarly the shape function for the N5 and N6 can be written as

For N5

$$N_5 = a_5\xi_1^2 + b_5\xi_2^2 + c_5\xi_3^2 + d_5\xi_1\xi_2 + e_5\xi_2\xi_3 + f_5\xi_1\xi_3 \quad \text{-----10}$$

Finding the unknown a5, b5, c5, d5, e5 and f5 and substituting in equation (10) gives

$$N_5 = 4\xi_2\xi_3 \quad \text{-------11}$$

For N6 $N_6 = a_6\xi_1^2 + b_6\xi_2^2 + c_6\xi_3^2 + d_6\xi_1\xi_2 + e_6\xi_2\xi_3 + f_6\xi_1\xi_3$ -----12

Finding the unknown a6, b6, c6, d6, e6 and f6 and substituting in equation (12) gives

$$N_6 = 4\xi_1\xi_3 \quad \text{-------13}$$

The shape function of the 6-noded triangular element are

$$N_1 = \xi_1(2\xi_1 - 1)$$

$$N_2 = \xi_2(2\xi_2 - 1)$$

$$N_3 = \xi_3(2\xi_3 - 1)$$

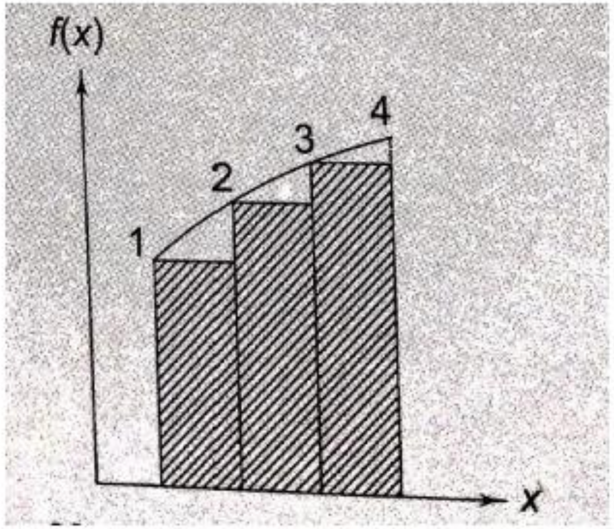
$$N_4 = 4\xi_1\xi_2$$

$$N_5 = 4\xi_2\xi_3$$

$$N_6 = 4\xi_1\xi_3$$

Numerical Integration

- The process of numerical integration is known as quadrature.
- Integration is basically a process of summation.



- Area under the curve $S = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$
- If $\Delta x \to 0, \&, n \to \infty$ but definite part

$$S = \sum_{i=1}^{n} f(x_i)\Delta x \qquad \text{---------(1)}$$

$$= \int f(x)dx$$

Newton Cote's Quadrature

If in equation (1) Δx is considered as a weight then $\int f(x)dx$ is considered to be approximately weighted sum of function values at discrete sampling points.

$$\int f(x)dx = \sum_{i=1}^{n} f(x_i)\Delta x$$

- Weight need to be equal and sampling point need not be equally spaced.
- In newton cotes, sampling point is taken as equally spaced and specified before finding weights.
- N-value of function define polynomial of degree n-1

Newton Cotes Quadrature

- N-point newton cotes formula

$$I = \int_{-1}^{1} f(\xi) d\xi = \sum_{i=1}^{n} w_i f(\xi_i) \qquad \text{------(2)}$$

- If the limits of integration are different then they can be transformed to -1 to +1 with the help of linear transformation.

$$x = \left(\frac{a+b}{2}\right) + \left(\frac{b-a}{2}\right)\xi \qquad \text{------(3)}$$

$$dx = \left(\frac{b-a}{2}\right) d\xi$$

2-Point Newton Cotes formula

- The expression for two point newton cotes formula for sampling point -1, +1 is written as

$$I = \int_{-1}^{+1} f(\xi) d\xi = w_1 f(-1) + w_2 f(+1) \text{ ------(4)}$$

- Two sampling point chosen -1 and +1 for polynomial

$$f(\xi) = a + b\xi$$

- At $\xi = -1,\quad f(-1) = a - b$ --------------(i)
- At $\xi = +1,\quad f(+1) = a + b$ --------------(ii)
- Solving equation (i) and (ii) for values of a and b

$$a = \frac{1}{2}[f(+1) + f(-1)] \qquad b = \frac{1}{2}[f(+1) - f(-1)]$$

We know, $f(\xi) = a + b\xi$

$$\int_{-1}^{1} f(\xi)d\xi = \int_{-1}^{1} (a + b\xi)d\xi = \left[a\xi + \frac{b\xi^2}{2} \right]_{-1}^{1} = 2(a) + 0 \qquad \text{-----(5)}$$

Substituting the value of 'a' in equation (5)

$$\int_{-1}^{1} f(\xi)d\xi = 2 \times \frac{1}{2}[f(-1) + f(1)]$$

$$\int_{-1}^{1} f(\xi)d\xi = [f(-1) + f(1)] \text{ ' } \quad w_1 = 1, w_2 = 1 \qquad \text{-----(6)}$$

This is called the trapezoidal rule.

3 point Newton Cotes Formula

Three sampling point are $\xi = -1, \xi = 0, \xi = +1$

The approximate function is $f(\xi) = a + b\xi + c\xi^2$

$$I = \int_{-1}^{+1} f(\xi) d\xi = w_1 f(-1) + w_2 f(0) + w_3 f(+1) \quad \text{-----(7)}$$

$$\int_{-1}^{1} f(\xi) d\xi = \int_{-1}^{1} (a + b\xi + c\xi^2) d\xi = 2a + \frac{2}{3}c \quad \text{-----(8)}$$

For finding the value of a, b, c, evaluating the function $f(\xi)$ at sampling point $\xi = -1, \xi = 0, \xi = +1$

$$f(-1) = a - b + c \qquad \text{------(i)}$$

$$f(0) = a \qquad \text{------(ii)}$$

$$f(-1) = a + b + c \qquad \text{------(iii)}$$

Solving equation (i), (ii) and (iii) for finding the values of c

$$c = \frac{f(-1) + f(1) - 2f(0)}{2}$$

Substituting the values of ‘a’ and ‘c’ in equation (8)

$$\int_{-1}^{1} f(\xi) d\xi = 2a + \frac{2}{3}c = 2f(0) + \frac{2}{3}\left[\frac{f(-1) + f(1) - 2f(0)}{2}\right]$$

$$= \frac{1}{3}\ f(-1) + 4f(0) + f(1) \qquad \text{-----(9)}$$

This is called Simpson’s (1/3) rule.

Thank You
For Your Attention